演習問題 102　● 2 階定数係数常微分方程式 (Ⅰ) ●

次の 2 階常微分方程式を，ラプラス変換を用いて解け。

(1) $y'' + y' + \frac{65}{4}y = 0$ ……①　$\left(y(0) = 0,\ y'(0) = 8\right)$

(2) $y'' + 3y' + \frac{9}{4}y = 0$ ……②　$\left(y(0) = 1,\ y'(0) = -\frac{1}{2}\right)$

ヒント！ 公式：$\mathcal{L}[y''(t)] = s^2Y(s) - sy(0) - y'(0)$，$\mathcal{L}[y'(t)] = sY(s) - y(0)$ や，$\mathcal{L}^{-1}[F(s-a)] = e^{at}\mathcal{L}^{-1}[F(s)]$ などを利用して解いていこう。

解答＆解説

(1) ①の両辺をラプラス変換すると，

$$\mathcal{L}\left[y''(t) + y'(t) + \frac{65}{4}y(t)\right] = 0$$

$$\underbrace{\mathcal{L}[y''(t)]}_{s^2Y(s)-sy(0)-y'(0)} + \underbrace{\mathcal{L}[y'(t)]}_{sY(s)-y(0)} + \frac{65}{4}\underbrace{\mathcal{L}[y(t)]}_{Y(s)} = 0$$

公式：
・$\mathcal{L}[y''] = s^2Y(s) - sy(0) - y'(0)$
・$\mathcal{L}[y'] = sY(s) - y(0)$

$$s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{8} + sY(s) - \underbrace{y(0)}_{0} + \frac{65}{4}Y(s) = 0$$

$$s^2Y(s) - 8 + sY(s) + \frac{65}{4}Y(s) = 0$$

$$\left(s^2 + s + \frac{65}{4}\right)Y(s) = 8$$

$\left(s + \frac{1}{2}\right)$でまとめることが，ポイント！

$$Y(s) = \frac{8}{s^2 + s + \frac{65}{4}} = \frac{8}{\left(s + \frac{1}{2}\right)^2 + 16} \quad \cdots\cdots\cdots\cdots ①'$$

①´の両辺をラプラス逆変換して，

公式：$\mathcal{L}^{-1}[F(s-a)] = e^{at}\mathcal{L}^{-1}[F(s)]$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{8}{\left(s + \frac{1}{2}\right)^2 + 16}\right] = e^{-\frac{1}{2}t}\mathcal{L}^{-1}\left[\frac{8}{s^2 + 16}\right]$$ より，

$$y(t) = e^{-\frac{1}{2}t}\cdot 2\cdot \underbrace{\mathcal{L}^{-1}\left[\frac{4}{s^2+4^2}\right]}_{\sin 4t}$$

公式：$\mathcal{L}^{-1}\left[\dfrac{a}{s^2+a^2}\right] = \sin at$

$$\therefore\ y(t) = 2e^{-\frac{1}{2}t}\sin 4t \quad\cdots\cdots\text{(答)}$$

(2) ②の両辺をラプラス変換すると，

$$\mathcal{L}\left[y''(t) + 3y'(t) + \frac{9}{4}y(t)\right] = 0$$

$$\underbrace{\mathcal{L}[y''(t)]}_{s^2Y(s)-sy(0)-y'(0)} + 3\underbrace{\mathcal{L}[y'(t)]}_{sY(s)-y(0)} + \frac{9}{4}\underbrace{\mathcal{L}[y(t)]}_{Y(s)} = 0$$

$$s^2Y(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{\left(-\frac{1}{2}\right)} + 3\{sY(s) - \underbrace{y(0)}_{1}\} + \frac{9}{4}Y(s) = 0$$

$$s^2Y(s) - s + \frac{1}{2} + 3sY(s) - 3 + \frac{9}{4}Y(s) = 0$$

$$\left(s^2 + 3s + \frac{9}{4}\right)Y(s) = s + \frac{5}{2}$$

$$Y(s) = \frac{s+\frac{5}{2}}{s^2+3s+\frac{9}{4}} = \frac{\left(s+\frac{3}{2}\right)+1}{\left(s+\frac{3}{2}\right)^2} \quad\cdots\cdots②'$$

$\left(s+\dfrac{3}{2}\right)$ でまとめることが，ポイント！

②´の両辺をラプラス逆変換して，

公式：$\mathcal{L}^{-1}[F(s-a)] = e^{at}\mathcal{L}^{-1}[F(s)]$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{\left(s+\frac{3}{2}\right)+1}{\left(s+\frac{3}{2}\right)^2}\right] = e^{-\frac{3}{2}t}\mathcal{L}^{-1}\left[\frac{s+1}{s^2}\right]$$

$$= e^{-\frac{3}{2}t}\left\{\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s}\right]}_{1} + \underbrace{\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]}_{t}\right\}$$

公式：$\mathcal{L}^{-1}\left[\dfrac{1}{s}\right] = 1,\ \mathcal{L}^{-1}\left[\dfrac{1}{s^2}\right] = t$

$$\therefore\ y(t) = (1+t)e^{-\frac{3}{2}t} \quad\cdots\cdots\text{(答)}$$